HAEF IB – FURTHER MATH HL

TEST 1

MATRICES – VECTOR SPACES

by Christos Nikolaidis

TOTAL SCORE (Sections A and B)	
Marks:	**/80**
Grade:	**%**

SECTION A: Without GDC

Marks:	**/40**

Name:_______________________________________

Date: **23 – 10 – 2017**

Questions

1. *[maximum mark: 7]*

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n\times n$ matrices. Show that

(a) $\det \boldsymbol{AB} = (\det \boldsymbol{A})(\det \boldsymbol{B})$ for $n = 2$ *[4 marks]*

(b) if $\boldsymbol{A}$ is invertible, the inverse of $\boldsymbol{A}$ is unique *[2 marks]*

(c) if $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible then $\boldsymbol{AB}$ is invertible *[1 mark]*

2. *[maximum mark: 10]*

Consider the system of simultaneous equations.

$$\begin{aligned} x - 2y - az &= b \\ 2x - y + 3z &= 2 \\ 3x + y + 2z &= -2 \end{aligned}$$

(a) Find the values of a and the values of b for which the system has

(i) a unique solution.

(ii) no solution.

(iii) infinitely many solutions. *[8 marks]*

(b) Find the general solution in case (a) (iii). *[2 marks]*

3. *[maximum mark: 13]*

Consider the simultaneous equations

$$-x - y + 6z = 1$$
$$x - 4z = 2$$
$$2x + 2y - 11z = 1$$

(a) Express the system in the form $AX = B$ and find A^{-1} *[5 marks]*

(b) **Hence** solve the system *[2 marks]*

(c) **Hence** write down the reduced row echelon form of the matrix $(A \mid B)$ *[1 mark]*

(d) Write down the rank of A and the rank of the augmented matrix $(A \mid B)$.

What can be said about the rows and the columns of $(A \mid B)$ *?* *[3 marks]*

(e) Express the column vector B as a linear combination of the columns of A *[2 marks]*

4. *[maximum mark: 10]*

Let $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ be non-zero vectors in R^n.

Show that

(a) if $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly independent **then** $\boldsymbol{u}+\boldsymbol{v}$, $\boldsymbol{v}+\boldsymbol{w}$ are linearly independent. *[3 marks]*

(b) the converse of (a) is not true, by using a counterexample. *[3 marks]*

Consider the statement

"If $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are linearly dependent
then $\boldsymbol{u}$ is a linear combination of $\boldsymbol{v}$ and $\boldsymbol{w}$"

(c) Explain by using a counterexample why the statement is false. *[3 marks]*

(d) State the correct version of the statement above *[1 mark]*

SECTION B: With GDC

Marks: /40

Name:____

Date: 23 – 10 – 2017

Questions

5. *[maximum mark: 6]*
Let

$$A = \begin{pmatrix} -5 & 2 & 3 \\ -2 & 1 & 3 \\ 5 & 1 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}$$

(a) Find $\det(BB^T)$. *[1 mark]*

(b) Find $\det(A^{2017})$. *[2 marks]*

(c) Solve the equation $AX - B = X$ *[3 marks]*

6. *[maximum mark: 3]*
Find the vectors X which satisfy

$$\begin{pmatrix} 2 & 3 & -1 \\ 3 & 2 & 1 \\ 1 & -1 & 2 \end{pmatrix} X = \begin{pmatrix} 2 \\ 7 \\ 5 \end{pmatrix}$$

7. *[maximum mark: 4]*
Let ***A*** be a 4×8 matrix. Show that the rank of ***A*** equals the nullity of ***A*** **if and only if** the rows of ***A*** are linearly independent.

8. *[maximum mark: 7]*

Let

$$V = \{\begin{pmatrix} x \\ x+y \\ y-x \end{pmatrix} \mid x, y \in R\} \qquad W = \{\begin{pmatrix} 0 \\ a \\ b \end{pmatrix} \mid a, b \in Q\}$$

(a) Show that V is a subspace of R^3 and find its dimension. *[4 marks]*

(b) Find the set $V \cap W$. Is it a subspace of R^3? Justify your answer. *[3 marks]*

9. *[maximum mark: 20]*

Let

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$$

(a) Show, by using mathematical induction, that for $n \in Z^+$

$$A^n = \begin{pmatrix} 2^{n+1} - 3^n & 2^{n+1} - 2 \times 3^n \\ 3^n - 2^n & 2 \times 3^n - 2^n \end{pmatrix}$$ *[6 marks]*

(b) Show, by using diagonalization, that for $n \in Z^+$

$$A^n = \begin{pmatrix} 2^{n+1} - 3^n & 2^{n+1} - 2 \times 3^n \\ 3^n - 2^n & 2 \times 3^n - 2^n \end{pmatrix}$$ *[7 marks]*

(c) Express A^2 in the form $aA + bI$. *[2 marks]*

(d) Express A^{-1} in the form $aA + bI$. *[2 marks]*

(e) Given that $A^n = aA + bI$, find expressions for a and b in terms of n *[3 marks]*