HAEF IB – FURTHER MATH HL

TEST 2

LINEAR ALGEBRA

by Christos Nikolaidis

Name:____________________________________

Date:________________

Marks:____**/100**

Grade: ______

Questions

1. *[maximum mark: 6]*

Let $V = \{ \begin{pmatrix} x \\ y \\ 0 \\ \frac{x+2y}{3} \end{pmatrix} : x, y \in R \}$.

(a) Show that V is a subspace of R^4. *[4 marks]*

(b) Find a basis of V and state its dimension. *[2 marks]*

2. *[maximum mark: 6]*

Show that

(a) $\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 13 \end{pmatrix}$ are linearly dependent vectors (by using the definition) *[3 marks]*

(b) $\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ form a basis for R^2. *[3 marks]*

3. *[maximum mark: 7]*

Let u, v be two vectors in R^n. Show that

(a) $u, v, 0$ are linearly dependent (by using the definition) *[2 marks]*

(b) $u+v$ and $u-v$ are linearly independent **if and only if** u and v are linearly independent *[5 marks]*

4. *[maximum mark: 6]*

(a) If U and V are subspaces of R^n show that $U \cap V$ is also a subspace of R^n. *[4 marks]*

(b) By selecting appropriate subspaces U and V of R^2 show $U \cup V$ is not necessarily a subspace of R^2. *[2 marks]*

5. *[maximum mark: 13]*

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2x2 matrix.

(a) Show that the characteristic polynomial is $P(\lambda) = \lambda^2 - (a+d)\lambda + \det A$ *[2 marks]*

(b) Show that the matrix A is a root of the corresponding matrix polynomial,

that is $P(A) = A^2 - (a+d)A + (\det A)I = 0$ (**Cayley–Hamilton** theorem) *[4 marks]*

Let now $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

(c) Use result (b) in order to express A^2 as a linear combination of A and I. *[2 marks]*

(d) Express A^{-1} in terms of A and I only. *[2 marks]*

(e) Express A^3 in terms of A and I only *[3 marks]*

6. *[maximum mark: 14]*

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \\ 1 & 2 & a \end{pmatrix}$

(a) Find the row rank of A. *[3 marks]*

(b) State the column rank of A. *[1 mark]*

(c) Describe the row space of A, in terms of a basis. *[3 marks]*

(d) Describe the column space when $a = 6$. *[2 marks]*

(e) Find the dimension of the null space of A and **hence** the null space. *[3 marks]*

(f) Find the dimension of the null space of A^T. *[2 marks]*

7. *[maximum mark: 6]*

Let W be a subspace of R^n. Show that

v_1, v_2, v_3 form a basis for W (i.e. v_1, v_2, v_3 are linearly independent and span W).

if and only if

any vector in W can be expressed **uniquely** as a linear combination of v_1, v_2, v_3

8. *[maximum mark: 9]*

Let $T : R^3 \to R^2$ given by $T\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} z \\ 2x \end{pmatrix}$

(a) Show that T is a linear transformation *[3 marks]*

(b) Find the kernel of *T*. *[2 marks]*

(c) Write down the range of *T*. *[2 marks]*

(d) Find the standard matrix corresponding to *T* *[2 marks]*

9. *[maximum mark: 13]*

Let $T : R^n \to R^m$ a linear transformation, where $\boldsymbol{0_n}$ and $\boldsymbol{0_m}$ are the corresponding zero vectors

(a) Show $T(\boldsymbol{0_n}) = \boldsymbol{0_m}$ *[3 marks]*

(b) Show that the kernel, ker*T,* of the transformation is a subspace of R^n. *[5 marks]*

(c) Show that T is one-to-one if and only if $\ker T = \{\boldsymbol{0_n}\}$. *[5 marks]*

10. *[maximum mark: 5]*

Let $T : R^2 \to R^2$ be a linear transformation given by $T : \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ 3x+4y \end{pmatrix}$

Find the image of the straight line $y = 2x$.

11. *[maximum mark: 15]*

Let $A = \begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$.

(a) Show that $A^2 = 4I$ and hence find formulas for

(i) A^{2k} (even powers)

(ii) A^{2k+1} (odd powers) *[4 marks]*

(b) Diagonalise the matrix *A* by using its eigenvalues and **hence** show that

$$A^n = 2^{n-2}\begin{pmatrix} 2[1+(-1)^n] & 1-(-1)^n \\ 4[1-(-1)^n] & 2[1+(-1)^n] \end{pmatrix}$$

[8 marks]

(c) Confirm that (a) and (b) give the same result for A^{2017} *[3 marks]*