HAEF IB – FURTHER MATH HL

TEST 1 – (P1: WITHOUT GDC)

MATRICES – VECTOR SPACES

by Christos Nikolaidis

Marks:____/40

Name:______________________________________

Date: **17 – 10 – 2016**

Questions

1. *[maximum mark: 4]*

Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be non-singular $n \times n$ matrices, $\boldsymbol{I}$ the $n \times n$ identity matrix and k a scalar. State which of the following statements are **correct**. For each incorrect statement, write down the correct version of the right hand side.

(a) $(\boldsymbol{A}+\boldsymbol{I})(\boldsymbol{A}-\boldsymbol{I}) = \boldsymbol{A}^2-\boldsymbol{I}$ *[1 mark]*

(b) $(\boldsymbol{A}+\boldsymbol{B})(\boldsymbol{A}-\boldsymbol{B}) = \boldsymbol{A}^2-\boldsymbol{B}^2$ *[1 mark]*

(c) $\boldsymbol{C}^2-\boldsymbol{AC} = \boldsymbol{C}(\boldsymbol{C}-\boldsymbol{A})$ *[1 mark]*

(d) $(\boldsymbol{AB})^T = \boldsymbol{A}^T\mathrm{B}^T$ *[1 mark]*

2. *[maximum mark: 5]*

Consider the system of simultaneous equations.

$$\begin{aligned} x+ay+bz &= c \\ ax-y+az &= b \\ bx+y+dz &= 3b \end{aligned}$$

Given that $(1,1,1)$ and $(8,5,-4)$ are solutions of the system, find

$$\det\begin{pmatrix} 1 & a & b \\ a & -1 & a \\ b & 1 & d \end{pmatrix}$$

3. *[maximum mark: 7]*

Let

$$A = \begin{pmatrix} -1 & -1 & 6 \\ 1 & 0 & -4 \\ 2 & 2 & -11 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

(a) Find A^{-1} by transforming $(A \mid I)$ into the reduced row echelon form. *[4 marks]*

(b) **Hence** solve $AX = B$. *[3 marks]*

4. *[maximum mark: 10]*

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors R^n, where $\boldsymbol{v} \neq \boldsymbol{0}$.

(a) $\boldsymbol{u}$ is linearly independent **if and only if** $\boldsymbol{u} \neq \boldsymbol{0}$ *[5 marks]*

(b) $\boldsymbol{u}$, $\boldsymbol{v}$ are linearly dependent **if and only if** $\boldsymbol{u}$ is a multiple of $\boldsymbol{v}$. *[5 marks]*

5. *[maximum mark: 14]*

Consider the system of equations $\boldsymbol{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -42 \end{pmatrix}$, where $\boldsymbol{T} = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & r \\ 3r & 0 & s \end{pmatrix}$.

(a) Find the solution of the system when $r = 0$ and $s = 3$.

[4 marks]

(b) The solution of the system is not unique.

(i) Show that $s = \frac{9}{2} r^2$.

(ii) When $r = 2$ and $s = 18$, show that the system can be solved, and find the general solution.

[10 marks]

FURTHER MATH.

TEST 1
PAPER 1
SOLUTIONS

1. (a) Correct

(b) $= A^2 - AB - BA + B^2$

(c) $= (C-A)C$

(d) $= B^T A^T$

2. Since it has two solutions it will have infinetely many solutions

Hence $\det A = 0$

3. (a)
$$\left(\begin{array}{ccc|ccc} -1 & -1 & 6 & 1 & 0 & 0 \\ 1 & 0 & -4 & 0 & 1 & 0 \\ 2 & 2 & -11 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 1 & -6 & -1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right)\begin{array}{l} -R_1 \\ R_2+R_1 \\ R_3+2R_1 \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & -6 & -1 & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right)\begin{array}{l} \\ -R_2 \\ \\ \end{array} \sim \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 11 & 0 & 6 \\ 0 & 1 & 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right)\begin{array}{l} R_1+6R_3 \\ R_2+2R_3 \\ \\ \end{array}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 1 & 4 \\ 0 & 1 & 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array}\right)\begin{array}{l} R_1-R_2 \\ \\ \\ \end{array} \qquad \text{Hence } A^{-1} = \begin{pmatrix} 8 & 1 & 4 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

(b) $AX=B \Rightarrow X=A^{-1}B \Rightarrow X=\begin{pmatrix} 8 & 1 & 4 \\ 3 & -1 & 2 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$\Rightarrow X=\begin{pmatrix} 14 \\ 3 \\ 3 \end{pmatrix}$

4. (a) "$\Leftarrow$" If $\vec{u}\neq\vec{0}$

$\lambda\vec{u}=0 \Rightarrow \lambda=0$ so $\vec{u}$ is lin. independent

"$\Rightarrow$" If $\vec{u}$ is lin. independent

then $\vec{u}\neq\vec{0}$. Otherwise, if $\vec{u}=\vec{0}$

$3\vec{u}=\vec{0}$ i.e $\vec{u}$ lin. dependent, contradiction.

(b) "$\Rightarrow$" Let $\vec{u},\vec{v}$ lin. dependent.

$\lambda_1\vec{u}+\lambda_2\vec{v}=\vec{0}$ where λ_1, λ_2 are not both 0.

if $\lambda_1=0$ $\lambda_2\vec{v}=0 \Rightarrow \vec{v}=0$ contradiction
so $\lambda_1\neq 0$ and $\vec{u}=-\frac{\lambda_2}{\lambda_1}\vec{v}$

"$\Leftarrow$" Let $\vec{u}$ be a multiple of $\vec{v}$, $\vec{u}=\lambda\vec{v}$

Then $\vec{u}-\lambda\vec{v}=0$ $(1\neq 0)$
So u,v lin. dependent.

5. (a) When $r=0$, $s=3$

$$T=\begin{pmatrix}-1 & 3 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{pmatrix} \qquad \begin{aligned}-x+3y&=4\\ 2y&=-2\\ 3z&=-42\end{aligned}$$

Hence $z=-14$, $y=-1$, $x=-7$

(b)(i) $\operatorname{Det} T=0$

$$\Rightarrow -\begin{vmatrix}2 & r\\ 0 & s\end{vmatrix}-3\begin{vmatrix}0 & r\\ 3r & s\end{vmatrix}=0$$

$$\Rightarrow -2s-3(-3r^2)=0$$

$$\Rightarrow -2s+9r^2=0 \Rightarrow s=\frac{9}{2}r^2$$

(ii) When $r=2$, $s=18$

$$\left(\begin{array}{ccc|c}-1 & 3 & 0 & 4\\ 0 & 2 & 2 & -2\\ 6 & 0 & 18 & -42\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & -3 & 0 & -4\\ 0 & 1 & 1 & -1\\ 1 & 0 & 3 & -7\end{array}\right)\begin{array}{l}-R_1\\ \frac{1}{2}R_2\\ \frac{1}{6}R_3\end{array}$$

$$\sim \left(\begin{array}{ccc|c}1 & -3 & 0 & -4\\ 0 & 1 & 1 & -1\\ 0 & 3 & 3 & -3\end{array}\right)_{R_3-R_1} \sim \left(\begin{array}{ccc|c}1 & -3 & 0 & -4\\ 0 & 1 & 1 & -1\\ 0 & 0 & 0 & 0\end{array}\right)_{R_3-R_2}$$

$z=\lambda$, $y=-1-\lambda$ $\quad x=-4+3y=-4+3(-1-\lambda)$
$=-7-3\lambda$.

HAEF IB – FURTHER MATH HL

TEST 1 – (P2: WITH GDC)

MATRICES – VECTOR SPACES

by Christos Nikolaidis

Marks:____/40

TOTAL SCORE

Marks: __ /80 (%)

Grade: ____

Name:____________________________________

Date: 17 – 10 – 2016

Questions

1. *[maximum mark: 5]*

Let

$$M=\begin{pmatrix}-5 & 2 & 3\\ -2 & 1 & 3\\ 5 & 1 & 4\end{pmatrix} \qquad N=\begin{pmatrix}a & 1 & 2\\ 3 & b & 4\\ 5 & 6 & c\end{pmatrix}$$

with $\det M=\det N$. Find $\det(MN)$.

[5 marks]

2. *[maximum mark: 4]*

Write down the **reduced row echelon form** of the augmented matrix corresponding to

$$\begin{aligned}2x+3y-z&=2\\3x+2y+z&=7\\x-y+2z&=5\end{aligned}$$

Hence find the general solution of the system.

3. *[maximum mark: 5]*

Let

$$\boldsymbol{u}=\begin{pmatrix}2\\1\\-3\end{pmatrix},\qquad \boldsymbol{v}=\begin{pmatrix}1\\4\\2\end{pmatrix},\qquad \boldsymbol{w}=\begin{pmatrix}3\\0\\1\end{pmatrix}$$

(a) Show that $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ are linearly independent. *[2 marks]*

(b) Express $\begin{pmatrix}1\\1\\1\end{pmatrix}$ as a linear combination of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$. *[3 marks]*

4. *[maximum mark: 8]*
Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) By observing the first powers of A (that is A^2, A^3, A^4) guess a formula for A^n in terms of n. *[3 marks]*

(b) Verify that your guess holds for the inverse matrix A^{-1} as well. *[2 marks]*

(c) Given that A^k has the form you guessed, show that A^{k+1} also has this form. *[3 marks]*

5. *[maximum mark: 8]*
Let

$$V = \left\{ \begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix} \mid x, y \in R \right\} \qquad W = \left\{ \begin{pmatrix} 0 \\ m \\ n \end{pmatrix} \mid m, n \text{ even integers} \right\}$$

(a) Show that V is a subspace of R^3. *[3 marks]*

(b) Investigate whether W is a subspace of R^3. *[3 marks]*

(c) Explain why $V \cup W$ is not a subspace of R^3. *[2 marks]*

6. *[maximum mark: 10]*
Suppose that

A is a singular $n \times n$ matrix (i.e. $\det A = 0$)
B is an $n \times 1$ matrix
O is the $n \times 1$ zero matrix.

(a) Write down the number of solutions of $AX = O$ *[1 mark]*

Let S_0 be a fixed solution of $AX = B$.

(b) Show that
S is also a solution of $AX = B$ **if and only if** $S - S_0$ is a solution of $AX = O$. *[6 marks]*

Let S_1 and S_2 be two distinct solutions of $AX = B$

(c) Show that $aS_1 + (1-a)S_2$ is also a solution of $AX = B$ for any $a \in R$. *[3 marks]*

FURTHER MATH

TEST 1
PAPER 2
SOLUTIONS

1. $\det M = 20$ (by GDC)

$\det(MN) = (\det M)(\det N) = 20 \cdot 20 = 400$

2. $$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 17/5 \\ 0 & 1 & -1 & -8/5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Hence
$$x = \frac{17}{5} - \lambda$$
$$y = -\frac{8}{5} + \lambda$$
$$z = \lambda$$

3. (a) $$\lambda_1\begin{pmatrix}2\\1\\3\end{pmatrix} + \lambda_2\begin{pmatrix}1\\4\\2\end{pmatrix} + \lambda_3\begin{pmatrix}3\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}2\lambda_1+\lambda_2+3\lambda_3\\ \lambda_1+4\lambda_2\\ 3\lambda_1+2\lambda_2+\lambda_3\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$ by GDC

(b) $$\begin{pmatrix}2\lambda_1+\lambda_2+3\lambda_3\\ \lambda_1+4\lambda_2\\ 3\lambda_1+2\lambda_2+\lambda_3\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$\lambda_1 = -3/49$
$\lambda_2 = 13/49$
$\lambda_3 = 2/7$

Hence $\begin{pmatrix}1\\1\\1\end{pmatrix} = -\frac{3}{49}u + \frac{13}{49}v + \frac{2}{7}w$

4. (a) $$A^n = \begin{pmatrix} 1 & 2n & 2n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) By GDC $$A^{-1} = \begin{pmatrix} 1 & -2 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which agrees with the formula above for $n=-1$

(c) $$A^{k+1} = A^k A = \begin{pmatrix} 1 & 2k & 2k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2k+2 & 2k+2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2(k+1) & 2(k+1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

5. (a)(i) $\vec{0} \in V$ (for $x=y=0$)

(ii) If $u = \begin{pmatrix} x_1 \\ y_1 \\ 2x_1+y_1 \end{pmatrix}$ $v = \begin{pmatrix} x_2 \\ y_2 \\ 2x_2+y_2 \end{pmatrix} \in V$

Then $u+v = \begin{pmatrix} x_1+x_2 \\ y_1+y_2 \\ 2(x_1+x_2)+(y_1+y_2) \end{pmatrix} \in V$

(iii) If $\lambda \in \mathbb{R}$ $u = \begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix} \in V$, $\lambda\vec{u} = \begin{pmatrix} \lambda x \\ \lambda y \\ 2\lambda x+\lambda y \end{pmatrix} \in V$.

(b) The third property fails

If $\lambda = \frac{1}{2}$ $u = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \in W$, $\lambda \vec{u} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \notin W$

So it is not a subspace

(c) $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} \in V \cup W$ but $\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 7 \end{pmatrix} \notin V \cup W$

6. (a) ∞-ly many solutions

(b) S is a solution of $AX = B$

$\Leftrightarrow AS = B$

$\Leftrightarrow AS = AS_0$

$\Leftrightarrow A(S - S_0) = 0$

$\Leftrightarrow S - S_0$ is a solution of $AX = 0$

(OR "$\Rightarrow$" and "$\Leftarrow$" separately)

(c) $AS_1 = B$ $AS_2 = B$ Then

$$\begin{aligned} A(aS_1 + (1-a)S_2) &= aAS_1 + (1-a)AS_2 \\ &= aB + (1-a)B \\ &= aB + B - aB \\ &= B. \end{aligned}$$