HAEF IB – FURTHER MATH HL

TEST 1 – (P1: WITHOUT GDC)

MATRICES – VECTOR SPACES

by Christos Nikolaidis

Marks:____/40

Name:______________________________________

Date: 17 – 10 – 2016

Questions

1. *[maximum mark: 4]*

Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be non-singular $n\times n$ matrices, $\boldsymbol{I}$ the $n\times n$ identity matrix and k a scalar. State which of the following statements are **correct**. For each incorrect statement, write down the correct version of the right hand side.

(a) $(\boldsymbol{A}+\boldsymbol{I})(\boldsymbol{A}-\boldsymbol{I}) = \boldsymbol{A}^2-\boldsymbol{I}$ *[1 mark]*

(b) $(\boldsymbol{A}+\boldsymbol{B})(\boldsymbol{A}-\boldsymbol{B}) = \boldsymbol{A}^2-\boldsymbol{B}^2$ *[1 mark]*

(c) $\boldsymbol{C}^2-\boldsymbol{AC} = \boldsymbol{C}(\boldsymbol{C}-\boldsymbol{A})$ *[1 mark]*

(d) $(\boldsymbol{AB})^{\boldsymbol{T}} = \boldsymbol{A}^{\boldsymbol{T}}\mathrm{B}^{\boldsymbol{T}}$ *[1 mark]*

2. *[maximum mark: 5]*

Consider the system of simultaneous equations.

$$x + ay + bz = c$$
$$ax - y + az = b$$
$$bx + y + dz = 3b$$

Given that (1,1,1) and (8,5,– 4) are solutions of the system, find

$$\det\begin{pmatrix} 1 & a & b \\ a & -1 & a \\ b & 1 & d \end{pmatrix}$$

3. *[maximum mark: 7]*

Let

$$A = \begin{pmatrix} -1 & -1 & 6 \\ 1 & 0 & -4 \\ 2 & 2 & -11 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

(a) Find A^{-1} by transforming $(A \mid I)$ into the reduced row echelon form. *[4 marks]*

(b) **Hence** solve $AX = B$. *[3 marks]*

4. *[maximum mark: 10]*

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two vectors R^n, where $\boldsymbol{v} \neq \boldsymbol{0}$.

(a) $\boldsymbol{u}$ is linearly independent **if and only if** $\boldsymbol{u} \neq \boldsymbol{0}$ *[5 marks]*

(b) $\boldsymbol{u}$, $\boldsymbol{v}$ are linearly dependent **if and only if** $\boldsymbol{u}$ is a multiple of $\boldsymbol{v}$. *[5 marks]*

5. *[maximum mark: 14]*

Consider the system of equations $\boldsymbol{T}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ -42 \end{pmatrix}$, where $\boldsymbol{T} = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & r \\ 3r & 0 & s \end{pmatrix}$.

(a) Find the solution of the system when $r = 0$ and $s = 3$.

[4 marks]

(b) The solution of the system is not unique.

(i) Show that $s = \dfrac{9}{2} r^2$.

(ii) When $r = 2$ and $s = 18$, show that the system can be solved, and find the general solution.

[10 marks]

HAEF IB – FURTHER MATH HL

TEST 1 – (P2: WITH GDC)

MATRICES – VECTOR SPACES

by Christos Nikolaidis

Marks:____/40

TOTAL SCORE

Marks: __ /80 (%)

Grade: ____

Name:______________________________________

Date: 17 – 10 – 2016

Questions

1. *[maximum mark: 5]*

Let

$$M = \begin{pmatrix} -5 & 2 & 3 \\ -2 & 1 & 3 \\ 5 & 1 & 4 \end{pmatrix} \qquad N = \begin{pmatrix} a & 1 & 2 \\ 3 & b & 4 \\ 5 & 6 & c \end{pmatrix}$$

with $\det M = \det N$. Find $\det(MN)$.

[5 marks]

2. *[maximum mark: 4]*

Write down the **reduced row echelon form** of the augmented matrix corresponding to

$$2x + 3y - z = 2$$
$$3x + 2y + z = 7$$
$$x - y + 2z = 5$$

Hence find the general solution of the system.

3. *[maximum mark: 5]*

Let

$$\boldsymbol{u} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}, \qquad \boldsymbol{v} = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \qquad \boldsymbol{w} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$$

(a) Show that $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are linearly independent. *[2 marks]*

(b) Express $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ as a linear combination of $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$. *[3 marks]*

4. *[maximum mark: 8]*
Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) By observing the first powers of A (that is A^2, A^3, A^4) guess a formula for A^n in terms of n. *[3 marks]*

(b) Verify that your guess holds for the inverse matrix A^{-1} as well. *[2 marks]*

(c) Given that A^k has the form you guessed, show that A^{k+1} also has this form. *[3 marks]*

5. *[maximum mark: 8]*
Let

$$V = \{\begin{pmatrix} x \\ y \\ 2x+y \end{pmatrix} \mid x, y \in R\} \qquad W = \{\begin{pmatrix} 0 \\ m \\ n \end{pmatrix} \mid m, n \text{ even integers}\}$$

(a) Show that V is a subspace of R^3. *[3 marks]*

(b) Investigate whether W is a subspace of R^3. *[3 marks]*

(c) Explain why $V \cup W$ is not a subspace of R^3. *[2 marks]*

6. *[maximum mark: 10]*
Suppose that

A is a singular $n \times n$ matrix (i.e. $\det A = 0$)

B is an $n \times 1$ matrix

O is the $n \times 1$ zero matrix.

(a) Write down the number of solutions of $AX = O$ *[1 mark]*

Let S_0 be a fixed solution of $AX = B$.

(b) Show that

S is also a solution of $AX = B$ **if and only if** $S - S_0$ is a solution of $AX = O$. *[6 marks]*

Let S_1 and S_2 be two distinct solutions of $AX = B$

(c) Show that $aS_1 + (1-a)S_2$ is also a solution of $AX = B$ for any $a \in R$. *[3 marks]*