MATH HL - OPTION CALCULUS

SOLUTIONS

2. DIFFERENTIAL EQUATIONS OF SEPARABE VARIABLES

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Past paper questions

1. $y = e^x - x^2 + C$ (A1)(A1)(A1)

$3 = e^0 - 0 + C \Rightarrow C = 2$ (M1) (A1)

$y = e^x - x^2 + 2$ (A1) (C6)

[6]

2.

Separating variables *(M1)*

$\int \frac{dy}{y^2} = \int 2x\, dx$ ***A1***

$-\frac{1}{y} = x^2 + C$ ***A1A1***

Note: The first ***A1*** above is for a correct LHS and the second ***A1*** is for a correct RHS that must include *C*.

Using $y(0) = 1$ gives $C = -1$ $\left(-\frac{1}{y} = x^2 - 1\right)$ ***M1***

$y = -\frac{1}{x^2 - 1}$ $\left(= \frac{1}{1 - x^2}\right)$ ***A1*** ***N0***

[6]

3. $x\frac{dy}{dx} - y^2 = 1, \Rightarrow x\frac{dy}{dx} = y^2 + 1$

Separating variables (M1)

$\frac{dy}{y^2 + 1} = \frac{dx}{x}$ A1

$\arctan y = \ln x + c$ A1A1

$y = 0, x = 2 \Rightarrow \arctan 0 = \ln 2 + c \Rightarrow -\ln 2 = c$ (A1)

$\arctan y = \ln x - \ln 2 = \ln\frac{x}{2}$

$y = \tan\left(\ln\frac{x}{2}\right)$ A1

[6]

4. $xy\frac{dy}{dx} = 1 + y^2 \Rightarrow \int \frac{y}{1 + y}dx = \int \frac{1}{x}dx$ (M1)

$\frac{1}{2}\ln(1 + y^2) = \ln x + \ln c$ (M1)

$1 + y^2 = kx^2$ $(k = c^2)$

$y = 0$ when $x = 2$, and so $1 = 4k$

Thus, $1 + y^2 = \frac{1}{4}x^2$ or $x^2 - 4y^2 = 4$. (A1) (C3)

[3]

5. Given $\frac{dx}{dt} = kx(5 - x)$

then $\frac{1}{x(5-x)}\frac{dx}{dt} = k$ (M1)

$\int \frac{1}{5x} + \frac{1}{5(5-x)}\,dx = \int k\,dt$ (A1)

$\frac{1}{5}\ln x - \frac{1}{5}\ln(5 - x) = kt$ **or** $\left(\frac{x}{5-x}\right)^{\frac{1}{5}} = Ae^{kt}$ **or** $\left(\frac{x}{5-x}\right) = Ae^{5kt}$ (A1) (C3)

[3]

6.

$\frac{dy}{y} = \frac{x\,dx}{x^2+1}$ *M1*

$\int \frac{dy}{y} = \ln y$ *A1*

$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)$ *A1*

EITHER

$\ln y = \frac{1}{2}\ln(x^2+1) + \ln C$

$\ln y = \ln C\sqrt{x^2+1}$

$1 = C\sqrt{2}$ for substituting $x = 1,\ y = 1$ *M1*

$C = \frac{1}{\sqrt{2}}$ *A1*

$y = \sqrt{\frac{x^2+1}{2}}$ *A1*

OR

$\ln y = \frac{1}{2}\ln(x^2+1) + A$

$\ln 1 = \frac{1}{2}\ln 2 + A$ for substituting $x = 1,\ y = 1$ *M1*

$A = -\frac{1}{2}\ln 2$ *A1*

$\ln y = \frac{1}{2}\ln(x^2+1) - \frac{1}{2}\ln 2\ \left(= \frac{1}{2}\ln(x^2+1) - 0.347\right)$ *A1*

$\left(\ln y = \ln\sqrt{\frac{x^2+1}{2}}\right)$

$\left(y = \sqrt{\frac{x^2+1}{2}}\right)$

[6]

7.

$$\int \frac{1}{y}\,\mathrm{d}y = \int \frac{4x}{(x+2)^2}\,\mathrm{d}x$$ *M1*

Let $u = x+2 \;\Rightarrow\; x = u-2$

$\mathrm{d}u = \mathrm{d}x$ *M1*

$$\ln y = \int \frac{4(u-2)}{u^2}\,\mathrm{d}u$$

$$= \int \frac{4}{u} - \frac{8}{u^2}\,\mathrm{d}u$$ *A1*

$\ln y = 4\ln u + 8u^{-1} + \mathrm{c}$ *A1*

$$\ln y = 4\ln(x+2) + \frac{8}{x+2} + \mathrm{c}$$ *A1*

$(-1, 1) \Rightarrow \mathrm{c} = -8$ *A1*

$$\Rightarrow \ln y = 4\ln(x+2) + \frac{8}{x+2} - 8$$ *N0*

[6]

8.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+y^2}{1+x^2}$$

$$\Rightarrow \int \frac{1}{1+y^2}\,\mathrm{d}y = \int \frac{1}{1+x^2}\,\mathrm{d}x$$ *M1*

$\Rightarrow \arctan y = \arctan x + k$ *A1*

$$\Rightarrow \arctan\sqrt{3} = \arctan\frac{\sqrt{3}}{3} + k$$

$$\Rightarrow \frac{\pi}{3} = \frac{\pi}{6} + k \Rightarrow k = \frac{\pi}{6}$$ *A1*

$$\Rightarrow \arctan y = \arctan x + \frac{\pi}{6}$$

$$\Rightarrow y = \tan\left(\arctan x + \frac{\pi}{6}\right)$$ *M1*

$$\Rightarrow y = \frac{x + \tan\frac{\pi}{6}}{1 - x\tan\frac{\pi}{6}}$$

$$\Rightarrow y = \frac{x + \frac{\sqrt{3}}{3}}{1 - x\frac{\sqrt{3}}{3}}$$ *A1*

$$\Rightarrow y = \frac{3x+\sqrt{3}}{3 - x\sqrt{3}}$$ *A1* *N0*

[6]

9. If $kx = mv\dfrac{\mathrm{d}v}{\mathrm{d}x}$ Then $\int kx\,\mathrm{d}x = \int mv\,\mathrm{d}v$ (using separation of variables) (M1)

$$\Rightarrow \frac{1}{2}kx^2 = \frac{1}{2}mv^2 + C$$ (A1)

When $x = 0$, $v = v_0$, therefore $C = -\dfrac{1}{2}mv_0^2$ Therefore $v^2 = v_0^2 + \dfrac{kx^2}{m}$

Therefore when $x = 2$, $v = \sqrt{v_0^2 + \dfrac{4k}{m}}$ (A1)

[3]

10. If A g is present at any time, then $\frac{\mathrm{d}A}{\mathrm{d}t} = ka$ where k is a constant.

Then, $\int \frac{\mathrm{d}A}{A} = k\int \mathrm{d}t$

$\Rightarrow \ln A = kt + c$

$\Rightarrow A = \mathrm{e}^{kt+c} = c_1\mathrm{e}^{kt}$

When $t = 0$, $c_1 = 50 \Rightarrow 48 = 50\mathrm{e}^{10k}$. (A1)

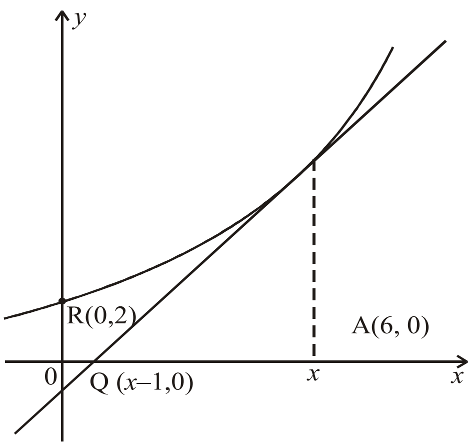
$\frac{\ln 0.96}{10} = k$ or $k = -0.00408(2)$ (A1)

For half life, $25 = 50\mathrm{e}^{kt} \Rightarrow \ln 0.5 = kt \Rightarrow t = \frac{10\ln 0.5}{\ln 0.96} = 169.8$.

Therefore, half-life = 170 years (3 s.f.) (A1) (C3)

[3]

11.



From the diagram,

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{1}$ (M1)(A1)

$\Rightarrow \int \frac{\mathrm{d}y}{y} = \int \mathrm{d}x \Rightarrow \ln y = x + c \Rightarrow y = \mathrm{e}^{x+c} = A\mathrm{e}^{x}$. (M1)(A1)(A1)

But R (0,2) lies on the curve and so $A = 2$. (A1)

Thus $y = 2\mathrm{e}^{x}$ (C6)

[6]

12. (a) Given $\frac{\mathrm{d}v}{\mathrm{d}t} = -kv \Leftrightarrow \int \frac{\mathrm{d}v}{v} = -k\int \mathrm{d}t$

$\Leftrightarrow \ln v = -kt + C$ (M1)

$\Leftrightarrow v = A\mathrm{e}^{-kt} (A = \mathrm{e}^{C})$

At $t = 0$, $v = v_0 \Rightarrow A = v_0 \Leftrightarrow v = v_0\mathrm{e}^{-kt}$ (A1) 2

(b) Put $v = \frac{v_0}{2}$ then $\frac{v_0}{2} = v_0\mathrm{e}^{-kt}$ (M1)

$\Leftrightarrow \frac{1}{2} = \mathrm{e}^{-kt} \Leftrightarrow \ln\frac{1}{2} = -kt$

$\Leftrightarrow t = \frac{\ln 2}{k}$ (A1) 2

Note: *Accept equivalent forms, e.g.* $t = \frac{\ln\frac{1}{2}}{-k}$

[4]